Instabilities, Point Attractors and Limit Cycles in a Inflationary Universe

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Abstract

We study the stability of a scalar inflaton field and analyze its point attractors in the phase space. We show that the value of the inflaton field in the vacuum is a bifurcation parameter and prove the possible existence of a limit cycle by using analytical and numerical arguments.

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The phenomenon of chaos[1], which has become very popular, rejuvenated interest in nonlinear dynamics, and the chaotic behaviour of classical field theories is currently subject of intensive research[2,3,4,5]. In this respect it is of great interest to investigate the existence of some precursor phenomena of chaotic motion. In particular nonlinear effects should lead to bifurcations which qualitatively change the system properties[6,7,8].

In this paper we study the stability of a scalar inflaton field[9,10,11] and analyze its bifurcation properties in the framework of the dynamical system theory[6,7,8]. The model we study is very schematic, so it can be seen as a toy model for classical nonlinear dynamics, with the attractive feature that it emerges form particle physics and cosmology.

It is generally believed that the universe, at a very early stage after the big bang, exhibited a short period of exponential expansion, the so–called inflationary phase[9,10,11]. In fact the assumption of an inflationary universe solves three major cosmological problems: the flatness problem, the homogeneity problem, and the formation of structure problem.

The Friedmann–Robertson–Walker metric[12] of a homogeneous and isotropic expanding universe is given by:

$$ds^2 = dt^2 - a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{1}$$

where $k = 1, -1$, or 0 for a closed, open, or flat universe, and $a(t)$ is the scale factor of the universe.

The evolution of the scale factor $a(t)$ is given by the Einstein equations:

$$\ddot{a} = -\frac{4\pi}{3}G(\rho + 3p)a,$$

$$(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi}{3} G\rho, \tag{2}$$

where ρ is the energy density of matter in the universe, and p its pressure. The gravitational constant $G = M_p^{-2}$ (with $\hbar = c = 1$), where $M_p = 1.2 \cdot 10^{19}$ GeV is the Plank mass, and $H = \dot{a}/a$ is the Hubble "constant", which in general is a function of time.

All inflationary models postulate the existence of a scalar field ϕ, the so–called inflation field, with Lagrangian[13]:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{3}$$

where the potential $V(\phi)$ depends on the type of inflation model considered. Here we choose a real field but also complex scalar can be used[11]. The scalar field, if minimally coupled to gravity, satisfies the equation[9,10,11]:

$$\Box\phi = \ddot{\phi} + 3(\frac{\dot{a}}{a})\dot{\phi} - \frac{1}{a^2}\nabla^2\phi = -\frac{\partial V}{\partial\phi}, \tag{4}$$

where $\Box$ is the covariant d'Alembertian operator. We suppose that in the universe there is only the inflaton field, so the Hubble "constant" H is related to the energy density of the field by:

$$H^2 + \frac{k}{a^2} = (\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi G}{3}[\frac{\dot{\phi}^2}{2} + \frac{(\nabla\phi)^2}{2} + V(\phi)]. \tag{5}$$

Immediately after the onset of inflation, the cosmological scale factor grows exponentially[9,10,11]. Thus the term $\nabla^2\phi/a^2$ is generally believed to be negligible and, if the inflaton field is sufficiently uniform (i.e. $\dot{\phi}^2$, $(\nabla\phi)^2 << V(\phi)$), we end up with a classical nonlinear scalar field theory in one dimension:

$$\ddot{\phi} + 3H(\phi)\dot{\phi} + \frac{\partial V}{\partial\phi} = 0, \tag{6}$$

where the Hubble "constant" H is an explicit function of ϕ:

$$H^2 = \frac{8\pi G}{3} V(\phi). \tag{7}$$

Most inflation scenarios are just formulated in this classical setting. In fact the quantization of the inflation scenario can be regarded to be still an open problem[14].

The classical inflaton dynamics can be studied in the framework of the dynamical system theory[6,7,8]. The second order equation of motion can be written as a system of two first order differential equations:

$$\dot{\vec{x}} = \vec{g}(\vec{x}) \tag{8}$$

where $\vec{x} = (\phi, \chi)$ is a point in the two dimensional phase space and $\vec{g} = (g_1, g_2)$ is given by:

$$g_1(\phi, \chi) = \chi, \quad g_2(\phi, \chi) = -3H(\phi)\chi - \frac{\partial V(\phi)}{\partial \phi}. \tag{9}$$

The system is non–conservative because the function

$$div(\vec{g}) = \frac{\partial g_1}{\partial \phi} + \frac{\partial g_2}{\partial \chi} = -3H(\phi) \tag{10}$$

is not identically zero. The fixed points of the system are those for which $g_1(\phi, \chi) = 0$ and $g_2(\phi, \chi) = 0$, i.e:

$$\chi = 0, \quad \frac{\partial V(\phi)}{\partial \phi} = 0. \tag{11}$$

The deviation $\delta\vec{x}(t) = \hat{\vec{x}}(t) - \vec{x}(t)$ from the two initially neighboring trajectories $\vec{x}$ and $\hat{\vec{x}}$ in the phase space satisfies the linearized equations of motion:

$$\frac{d}{dt}\delta\vec{x}(t) = \Gamma(t)\delta\vec{x} \tag{12}$$

where $\Gamma(t)$ is the stability matrix:

$$\Gamma(t) = \begin{pmatrix} 0 & 1 \\ -\frac{\partial^2 V}{\partial \phi^2} - 3\chi \frac{\partial H}{\partial \phi} & -3H(\phi) \end{pmatrix}. \qquad (13)$$

At least if an eigenvalue of $\Gamma(t)$ is real the separation of the trajectories grows exponentially and the motion is unstable. Imaginary eigenvalues correspond to stable motion. In the limit of time that goes to infinity the eigenvalues of the stability matrix are the Lyapunov exponent[1]. It is well known that for two–dimensional dynamical system the Lyapunov exponent can not be positive and so the system is not chaotic, i.e. there is not global instability.

However, we can be assured that the universe is crowded with many interacting fields of which the inflaton is but one[11]. The nonlinear nature of these interactions can result in a complex chaotic evolution of the universe and the local instability of the inflaton field is a precursor phenomenon of chaotic motion. Following the Toda criterion[15], we assume that the time dependence can be eliminated, i.e. $\Gamma(\phi(t), \chi(t)) = \Gamma(\phi, \chi)$.

The eigenvalues of the stability matrix are given by:

$$\sigma_{1,2} = -\frac{3}{2} H(\phi) \pm \frac{1}{2} \sqrt{9H^2(\phi) - 4\frac{\partial^2 V}{\partial \phi^2} - 12\chi \frac{\partial H}{\partial \phi}}. \qquad (14)$$

The pair of eigenvalues become real and there is exponential separation of neighboring trajectories, i.e. unstable motion, if:

$$\frac{\partial^2 V}{\partial \phi^2} + 3\chi \frac{\partial H}{\partial \phi} < 0. \qquad (15)$$

Particularly when $\chi = 0$, e.g. the fixed points, we obtain local instability when:

$$\frac{\partial^2 V}{\partial \phi^2} < 0, \qquad (16)$$

i.e. for negative curvature of the potential energy. The fixed points are stable if they are point of local minimum of $V(\phi)$ and unstable if are points of local maximum.

As stressed at the beginning the potential $V(\phi)$ depends on the type of inflation model considered, and it is usually some kind of double–well potential. We choose a symmetric double–well potential:

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2, \tag{17}$$

where $\pm v$ are the values of the inflaton field in the vacuum, i.e. the points of minimal energy of the system.

We observe that the inflaton field value in the vacuum v is a bifurcation parameter. Bifurcation is used to indicate a qualitative change in the features of the system under the variation of one or more parameters on which the system depends[6,7,8]. First of all we consider the case $v = 0$, i.e. $V(\phi) = (\lambda/4)\phi^4$. In this situation there is only one fixed point $(\phi^* = 0, \chi^* = 0)$ which is a stable one being:

$$\frac{\partial^2 V}{\partial \phi^2} = 3\lambda\phi^2 \geq 0. \tag{18}$$

The fixed point $(\phi^* = 0, \chi^* = 0)$ is a point attractor, as shown in Fig. 1 where we use a fourth order Runge–Kutta method[16] to compute the classical trajectories in the phase space.

Instead for $v \neq 0$ there are three fixed points:

$$(\phi^* = 0, \chi^* = 0), \quad (\phi^* = v, \chi^* = 0), \quad (\phi^* = -v, \chi^* = 0), \tag{19}$$

and the condition for the instability becomes:

$$-\frac{v}{\sqrt{3}} < \phi < \frac{v}{\sqrt{3}}. \tag{20}$$

Obviously $(\phi^* = 0, \chi^* = 0)$ is an unstable fixed point, and in particular a saddle point because the stability matrix has real and opposite eigenvalues. On the other hand $(\phi^* = \pm v, \chi^* = 0)$ are stable fixed points.

Form eq. (7) we find four possible functions for the Hubble "constant":

$$H(\phi) = \pm\gamma|\phi^2 - v^2|, \tag{21}$$

but also:

$$H(\phi) = \pm\gamma(\phi^2 - v^2), \tag{22}$$

where $\gamma = \sqrt{2\pi G\lambda/3}$ is the dissipation parameter. The choice of the Hubble function is crucial for the dynamical evolution of the system.

In certain non–conservative systems we could find closed trajectories or limit cycles toward which the neighboring trajectories spiral on both sides. It is sometime possible to know that no limit cycle exist and the Bendixson criterion[17], which establishes a condition for the non–existence of closed trajectories, is useful in some cases. Bendixson criterion is as follows: if $div(\vec{g})$ is not zero and does not change its sign within a domain D of the phase space, no closed trajectories can exist in that domain.

In our case we have $div(\vec{g}) = -3H(\phi)$, and so the presence of periodic orbit is related to the sign of $H(\phi)$. If $H(\phi) = \gamma|\phi^2 - v^2|$ we do not find periodic orbits and the inflaton field goes to one of its two stable fixed points, which are points attractors (see Fig. 2). The vacuum is degenerate but if we choose an initial condition around the saddle point there is a dynamical symmetry breaking[13] towards the positive v or negative $-v$ value of the inflaton field in the vacuum. This symmetry breaking is unstable because neighbour initial conditions can go in different point attractors.

Instead, if we choose $H(\phi) = \gamma(\phi^2 - v^2)$ the numerical calculations of Fig. 3 show that exists a limit cycle, the two stable fixed points are not point attractors, and the inflaton field oscillates forever. Obviously more large is v more large is the limit cycle.

In conclusion, we have studied the stability of a scalar inflaton field. With a symmetric double–well self energy the value of the inflaton field in the vacuum is a bifurcation parameter which changes dramatically the phase space structure. The main point is that for some functional solutions of the Hubble "constant" the system goes to a limit cycle, i.e. to a periodic orbit.

The inflaton field is not chaotic but its local instability can give rise to a complex chaotic evolution of the universe due to its nonlinear interactions with other fields. In the future it will be very interesting to study these effects which can perhaps lead to some observable implications like a fractal pattern in the spectrum of density fluctuations.

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Figure Captions

Figure 1: Inflaton field *vs* time (up), and its phase space trajectory (down), with $\gamma = 1/2$, $\lambda = 1$ and $v = 0$. Initial conditions: $\phi = 0$ and $\chi = \dot{\phi} = 1/2$.
Figure 2: Inflaton field *vs* time (up), and its phase space trajectory (down), for $H(\phi) = \gamma|\phi^2 - v^2|$ with $\gamma = 1/2$, $\lambda = 1$ and $v = 1$. Initial conditions: $\phi = 0$ and $\chi = \dot{\phi} = 1/2$.
Figure 3: Inflaton field *vs* time (up), and its phase space trajectory (down), for $H(\phi) = \gamma(\phi^2 - v^2)$ with $\gamma = 1/2$, $\lambda = 1$ and $v = 1$. Initial conditions: $\phi = 0$ and $\chi = \dot{\phi} = 1/2$.